

Electromagnetic wave propagation through position dependent permeability and permittivity

S. Habib Mazharimousavi,^{*} Ashkan Roozbeh,[†] and M. Halilsoy[‡]

Physics Department, Eastern Mediterranean University, G. Magusa north Cyprus, Mersin 10 Turkey

We use Maxwell's equations in a sourceless, non-uniform medium with position dependent, continuous permeability μ and permittivity ϵ to study the wave propagation. The general form of the wave equation is derived and by virtue of some physical assumptions, including μ and ϵ as functions of z , the equation has significantly been simplified. Finally by introducing a smooth step dielectric variable we solve the wave equation in the corresponding medium which is in conform with the well known results. Exact double-layer solution has been given in terms of the Heun functions.

I. INTRODUCTION

Having the electric permittivity of a system to be an isotropic and continuous spatial function has numerous applications in chemistry, biophysics and electronics [1–7]. As an example, in heavy doped regions the dielectric constant changes with the density of impurity and so with the position. Such a region can be found in bipolar transistors, $p-n$ junctions and solar cells [2]. It is also common that permittivity is assumed to be an isotropic spatial function that changes continuously in solvent region that allows us to formulate a computational scheme [3]. Other position-dependent medium can be seen in a biological membrane, like a lipid bilayer surrounded by water. For this system permittivity changes from a large value in the surrounding water to a lower value in the bilayer [4]. Also in electric double layers (EDL) due to ion accumulation and hydration in the region we face with permittivity variation, with effects on electric potential and interaction pressure between surfaces [5]. The main concern in such studies is the behaviour of a non-uniform mixed medium (a chemical solution or a $p-n$ junction) upon an external static electric field using the Poisson-Boltzmann equation. This differential equation (linear or non-linear form) can describe the electrostatic effects extensively, ranging from a bimolecular system [6] to an electrolyte solution [7]. The most regular form of this equation can be written as

$$\nabla \cdot [\epsilon(\mathbf{r}) \nabla \psi(\mathbf{r})] = -\rho_f(\mathbf{r}) - \sum_i c_i^\infty z_i q \lambda(\mathbf{r}) \exp \left[\frac{-z_i q \psi(\mathbf{r})}{\kappa_B T} \right]. \quad (1)$$

In this equation $\epsilon(\mathbf{r})$ is the position-dependent dielectric, $\psi(\mathbf{r})$ is the electrostatic potential and $\rho_f(\mathbf{r})$ represents the charge density of the medium. Further, z_i and c_i^∞ show the charge and the concentration of ions, T is the temperature, κ_B is the Boltzmann constant and $\lambda(\mathbf{r})$ is a factor that depends on the accessibility of a position to ions in the medium.

In the present work, we study the wave propagation inside a system with the electric permittivity $\epsilon(\mathbf{r})$ and the magnetic permeability $\mu(\mathbf{r})$, for some isotropic and continuous spatial (i.e. position-dependent) functions. Without external sources (i.e., $\rho_{free} = 0$, $\mathbf{J}_{free} = 0$) the wave equation for the electric component of the electromagnetic wave (EMW) propagating in the medium with $\epsilon = \epsilon(\mathbf{r})$ and $\mu = \mu(\mathbf{r})$ is found to be

$$\nabla^2 \mathbf{E} - \epsilon \mu \frac{\partial^2 \mathbf{E}}{\partial t^2} = -(\nabla \tilde{\epsilon} \cdot \nabla) \mathbf{E} - (\mathbf{E} \cdot \nabla) \nabla \tilde{\epsilon} - \nabla (\tilde{\epsilon} + \tilde{\mu}) \times (\nabla \times \mathbf{E}) \quad (2)$$

in which $\tilde{\epsilon} = \ln \epsilon$ and $\tilde{\mu} = \ln \mu$. A similar equation can be written for magnetic component of the EMW but we shall find \mathbf{B} using one of the Maxwell's equation. It is obvious that with ϵ and μ constants Eq. (2) reduces to the well known wave equation. A general approach toward the solution for Eq. (2) may not be possible due to the complicated form of the right hand side of the equation but by some simplifications we shall find exact analytical solution for this wave-type equation. We shall assume that the ϵ and μ vary only in one direction which is also the direction of propagation. This direction throughout the paper will be z -direction.

^{*}Electronic address: habib.mazhari@emu.edu.tr

[†]Electronic address: ashkan.physics@gmail.com

[‡]Electronic address: mustafa.halilsoy@emu.edu.tr

Organization of the paper is as follows. In Sec. II we find the form of the wave equation in a general system of coordinates and through some specifications we simplify the wave equation. In Sec. III we introduce the smooth step dielectric constant and solve the wave equation accordingly. Sec. IV solves the problem of smooth double layers. Our conclusion is presented in Sec. V.

II. THE WAVE EQUATION

We start with the sourceless ($\rho_{free} = 0, \mathbf{J}_{free} = 0$) Maxwell's equations in a medium with position-dependent permittivity ϵ and permeability μ which are given by

$$\nabla \cdot \mathbf{B} = 0, \quad \nabla \cdot \mathbf{D} = 0, \quad \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \quad \text{and} \quad \nabla \times \mathbf{H} = \frac{\partial \mathbf{D}}{\partial t}. \quad (3)$$

Herein $\mathbf{D} = \epsilon \mathbf{E}$ is the displacement vector, \mathbf{E} is the electric field, $\mathbf{H} = \frac{1}{\mu} \mathbf{B}$ is the auxiliary magnetic field and \mathbf{B} is the magnetic field. Combining the Maxwell's equations and after some manipulations one gets Eq. (2) and upon taking into account that $\epsilon = \epsilon(z)$ and $\mu = \mu(z)$ it becomes

$$\nabla^2 \mathbf{E} - \epsilon \mu \frac{\partial^2 \mathbf{E}}{\partial t^2} = - \left(\tilde{\epsilon}' \frac{\partial}{\partial z} \right) \mathbf{E} - (\mathbf{E} \cdot \nabla) \tilde{\epsilon}' \hat{k} - (\tilde{\epsilon}' + \tilde{\mu}') \hat{k} \times (\nabla \times \mathbf{E}), \quad (4)$$

in which a prime ' denotes $\frac{d}{dz}$. Our further simplification is to consider that the electric and magnetic components of the EMW vary in z direction only. This assumption is due to the symmetry of the medium which is physically acceptable. Our latter equation then reduces effectively into the following 1-dimensional three equations for the electric components

$$\left(\frac{\partial^2}{\partial z^2} - \mu \epsilon \frac{\partial^2}{\partial t^2} \right) E_x = \tilde{\mu}' \frac{\partial E_x}{\partial z}, \quad (5)$$

$$\left(\frac{\partial^2}{\partial z^2} - \mu \epsilon \frac{\partial^2}{\partial t^2} \right) E_y = \tilde{\mu}' \frac{\partial E_y}{\partial z}, \quad (6)$$

$$\left(\frac{\partial^2}{\partial z^2} - \mu \epsilon \frac{\partial^2}{\partial t^2} \right) E_z = -\tilde{\epsilon}' E_z' - E_z \tilde{\epsilon}''. \quad (7)$$

Next we consider

$$\mathbf{E}(\mathbf{r}, t) = \bar{\mathbf{E}}(z) e^{i\omega t} \quad (8)$$

in which ω is the angular frequency of the wave. A substitution in (5)-(8) yields:

$$\left(\frac{d^2}{dz^2} + \mu \epsilon \omega^2 \right) \bar{E}_i(z) = \frac{\mu'}{\mu} \frac{d\bar{E}_i(z)}{dz} \quad (9)$$

for $i = x, y$. To have the third equation (i.e. Eq. (7)) satisfied we set $E_z = 0$ (i.e. no longitudinal component) which is equivalent with the propagation of the wave in z -direction. Having symmetry with respect to x and y one can always rotate the system of coordinates in z direction such that $\bar{\mathbf{E}}(z)$ aligns with one of the coordinates (say x). Hence we are left with only one equation in x -direction which is expressed as

$$\left(\frac{d^2}{dz^2} + \mu \epsilon \omega^2 \right) \bar{E}_x(z) = \frac{\mu'}{\mu} \frac{d\bar{E}_x(z)}{dz} \quad (10)$$

and the other two components of the electric field are zero i.e., $\mathbf{E}(\mathbf{r}, t) = \hat{\mathbf{x}}\bar{E}_x(z)e^{i\omega t}$. To do further investigation one must know the form of μ and ϵ in terms of z . For instance, with $\mu = \mu_0 = \text{const.}$ and $\epsilon = \epsilon_0 = \text{const.}$ one finds

$$\left(\frac{d^2}{dz^2} + \frac{\omega^2}{c^2}\right)\bar{E}_x = 0 \quad (11)$$

which admits

$$\bar{E}_x = e^{\mp ikz} \quad (12)$$

where $k = \frac{\omega}{c}$ and the plane wave is propagating in $\pm z$ direction. Our final remark in this section is on the form of the wave equation: if one considers $\mu' = 0$ reduces to the form of the standard wave equation, but owing to the form of $\epsilon = \epsilon(z)$ its solution differs from the standard wave equation.

III. SMOOTH STEP DIELECTRIC CONSTANT

As an example let's consider $\mu = K_m\mu_0 = \text{const.}$ and $\epsilon = K_e(z)\epsilon_0$, where $K_e(z)$ stands for a smooth function of z given by

$$K_e(z) = K_2 - \frac{\Delta K}{4}(1 - \tanh(az))^2, \quad (13)$$

in which a is a positive real constant, $\Delta K = K_2 - K_1$, $K_2 = \lim_{z \rightarrow \infty} K_e(z)$ and $K_1 = \lim_{z \rightarrow -\infty} K_e(z)$. The wave equation (9) becomes

$$\left(\frac{d^2}{dz^2} + \frac{\omega^2}{c^2}K_m K_e(z)\right)\bar{E}_x(z) = 0, \quad (14)$$

which after defining the following new parameters

$$\kappa^2 = \frac{\omega^2}{c^2}K_m K_1, \quad \nu^2 = \frac{\omega^2}{c^2}K_m K_2 \quad (15)$$

and considering a new variable

$$\xi = -e^{-2az} \quad (16)$$

together with a redefinition of the electric field

$$\bar{E}_x(z) = (-\xi)^{-i\nu} F(\xi), \quad (17)$$

it turns (with $' = \frac{d}{d\xi}$) into

$$\xi F'' + (1 - 2i\nu)F' + \frac{1}{4a^2} \left[\frac{(1 - 4a^2)\nu^2}{\xi} + \frac{\nu^2 - \kappa^2}{1 - \xi} - \frac{\nu^2 - \kappa^2}{(1 - \xi)^2} \right] F = 0. \quad (18)$$

Having singularities at $\xi = 0$ and $\xi = 1$ suggests to replace further

$$F(\xi) = \xi^\sigma (\xi - 1)^\rho G(\xi) \quad (19)$$

which after some manipulation and choices

$$\sigma = \frac{i\nu(2a - 1)}{2a} \quad (20)$$

and

$$\rho = \frac{1}{2} \left(1 - \frac{1}{a} \sqrt{a^2 + \nu^2 - \kappa^2} \right) \quad (21)$$

it reduces to the following Hypergeometric differential equation (HDE) [8]

$$\xi(\xi-1)G'' + \left[\frac{i\nu-a}{a} - \left(\frac{i\nu-2a}{a} + \frac{1}{a}\sqrt{a^2+\nu^2-\kappa^2} \right) \xi \right] G' - \frac{i\nu-a}{2a^2} \left(a - \sqrt{a^2+\nu^2-\kappa^2} \right) G = 0. \quad (22)$$

Comparing with the standard form of the Hypergeometric DE

$$\xi(\xi-1)G'' + [(\alpha+\beta+1)\xi-\gamma]G' + \alpha\beta G = 0, \quad (23)$$

one finds

$$\alpha = \frac{1}{2a} \left[a - \sqrt{a^2+\nu^2-\kappa^2} - i(\nu+\kappa) \right], \quad (24)$$

$$\beta = \frac{1}{2a} \left[a - \sqrt{a^2+\nu^2-\kappa^2} - i(\nu-\kappa) \right], \quad (25)$$

and

$$\gamma = \frac{a-i\nu}{a}. \quad (26)$$

The general solution for the above HDE can be written as

$$G = C_1 F(\alpha, \beta; \gamma; \xi) + C_2 \xi^{1-\gamma} F(\alpha-\gamma+1, \beta-\gamma+1; 2-\gamma; \xi) \quad (27)$$

in which C_1 and C_2 are two integration constants. Going to the original variables now, the general solution for the Electric field is given by

$$\begin{aligned} \bar{E}_x(z) = & \tilde{C}_1 (-\xi)^{-\frac{i\nu}{2a}} (1-\xi)^\rho F(\alpha, \beta; \gamma; \xi) + \\ & \tilde{C}_2 (-\xi)^{\frac{i\nu}{2a}} (1-\xi)^\rho F(\alpha-\gamma+1, \beta-\gamma+1; 2-\gamma; \xi), \end{aligned} \quad (28)$$

where $\tilde{C}_1 = C_1(-1)^{\sigma+\rho}$ and $\tilde{C}_2 = C_2(-1)^{\sigma+\rho+1-\gamma}$. Let us consider that the wave comes from $z = -\infty$ and goes toward $z = +\infty$. Also we recall that $\lim_{z \rightarrow \infty} K_e(z) = K_2 = \text{cons.}$ which implies that $\lim_{z \rightarrow \infty} \bar{E}_x(z) \sim e^{i\frac{\omega}{c}\sqrt{K_m K_2}z} = e^{i\nu z}$. Once $z \rightarrow +\infty$ it is clear that $\xi = -e^{-2az} \rightarrow 0$, and upon knowing that $F(\alpha, \beta; \gamma; 0) = 1$, makes the limit of the electric field to be

$$\lim_{z \rightarrow \infty} \bar{E}_x(z) = \tilde{C}_1 e^{i\nu z} + \tilde{C}_2 e^{-i\nu z}. \quad (29)$$

This suggests that for this choice we must set $\tilde{C}_2 = 0$, which casts the solution into

$$\bar{E}_x(z) = \tilde{C}_1 (-\xi)^{-\frac{i\nu}{2a}} (1-\xi)^\rho F(\alpha, \beta; \gamma; \xi), \quad (30)$$

so that

$$\lim_{z \rightarrow \infty} \bar{E}_x(z) = \tilde{C}_1 e^{i\nu z} = E_{02} e^{i\nu z}. \quad (31)$$

Herein we consider the amplitude of the transmitted wave as E_{02} .

The limit of $z \rightarrow -\infty$ (and consequently $\xi \rightarrow -\infty$) can be found once we apply the following properties for the Hypergeometric functions

$$\begin{aligned} F(\alpha, \beta; \gamma; \xi) = & \frac{\Gamma(\gamma)\Gamma(\beta-\alpha)}{\Gamma(\beta)\Gamma(\gamma-\alpha)} (-1)^\alpha \xi^{-\alpha} F\left(\alpha, \alpha+1-\gamma; \alpha+1-\beta; \frac{1}{\xi}\right) + \\ & \frac{\Gamma(\gamma)\Gamma(\alpha-\beta)}{\Gamma(\alpha)\Gamma(\gamma-\alpha)} (-1)^\beta \xi^{-\beta} F\left(\beta, \beta+1-\gamma; \beta+1-\alpha; \frac{1}{\xi}\right). \end{aligned} \quad (32)$$

Hence

$$\lim_{\substack{z \rightarrow -\infty \\ \xi \rightarrow -\infty}} F(\alpha, \beta; \gamma; \xi) = \lim_{\substack{z \rightarrow -\infty \\ \xi \rightarrow -\infty}} \frac{\Gamma(\gamma)\Gamma(\beta-\alpha)}{\Gamma(\beta)\Gamma(\gamma-\alpha)} (-1)^\alpha \xi^{-\alpha} + \frac{\Gamma(\gamma)\Gamma(\alpha-\beta)}{\Gamma(\alpha)\Gamma(\gamma-\alpha)} (-1)^\beta \xi^{-\beta}, \quad (33)$$

which upon (28), one finds

$$\lim_{\substack{z \rightarrow -\infty \\ \xi \rightarrow -\infty}} \bar{E}_x(z) = \tilde{C}_1 \left(\frac{\Gamma(\gamma)\Gamma(\beta-\alpha)}{\Gamma(\beta)\Gamma(\gamma-\alpha)} (-\xi)^{-\alpha-\frac{i\nu}{2a}+\rho} + \frac{\Gamma(\gamma)\Gamma(\alpha-\beta)}{\Gamma(\alpha)\Gamma(\gamma-\alpha)} (-\xi)^{-\beta-\frac{i\nu}{2a}+\rho} \right). \quad (34)$$

Now one can show that

$$-\alpha - \frac{i\nu}{2a} + \rho = +i\frac{\kappa}{2a}, \quad (35)$$

$$-\beta - \frac{i\nu}{2a} + \rho = -i\frac{\kappa}{2a} \quad (36)$$

which finally yields

$$\lim_{\substack{z \rightarrow -\infty \\ \xi \rightarrow -\infty}} \bar{E}_x(z) = \tilde{C}_1 \left(\frac{\Gamma(\gamma)\Gamma(\beta-\alpha)}{\Gamma(\beta)\Gamma(\gamma-\alpha)} e^{-i\mu z} + \frac{\Gamma(\gamma)\Gamma(\alpha-\beta)}{\Gamma(\alpha)\Gamma(\gamma-\alpha)} e^{i\mu z} \right) = E'_{01} e^{-i\mu z} + E_{01} e^{i\mu z}. \quad (37)$$

Note that

$$E_{01} = \frac{\Gamma(\gamma)\Gamma(\alpha-\beta)}{\Gamma(\alpha)\Gamma(\gamma-\alpha)} E_{02} \quad (38)$$

is the amplitude of the transmitted wave for $z \rightarrow \infty$ and

$$E'_{01} = \frac{\Gamma(\gamma)\Gamma(\beta-\alpha)}{\Gamma(\beta)\Gamma(\gamma-\alpha)} E_{02} \quad (39)$$

is the amplitude of the reflected wave for $z \rightarrow -\infty$. Having these definitions together with the above equations one can introduce the reflection and transmission coefficients of the wave as

$$R = \frac{E'_{01}}{E_{01}} = \frac{\Gamma(\alpha)\Gamma(\beta-\alpha)}{\Gamma(\beta)\Gamma(\alpha-\beta)} \quad (40)$$

and

$$T = \frac{E_{02}}{E_{01}} = \frac{\Gamma(\alpha)\Gamma(\gamma-\alpha)}{\Gamma(\gamma)\Gamma(\alpha-\beta)}. \quad (41)$$

In the limit of a sharp step dielectric i.e., $a \rightarrow \infty$ one finds

$$\lim_{a \rightarrow \infty} R = \lim_{a \rightarrow \infty} \frac{\Gamma(\alpha)\Gamma(\beta-\alpha)}{\Gamma(\beta)\Gamma(\alpha-\beta)} = \frac{\kappa-\nu}{\kappa+\nu} = \frac{\sqrt{K_m K_1} - \sqrt{K_m K_2}}{\sqrt{K_m K_1} + \sqrt{K_m K_2}} = \frac{n_1 - n_2}{n_1 + n_2} \quad (42)$$

$$\lim_{a \rightarrow \infty} T = \lim_{a \rightarrow \infty} \frac{\Gamma(\gamma)\Gamma(\alpha-\beta)}{\Gamma(\alpha)\Gamma(\gamma-\alpha)} = \frac{2\kappa}{\kappa+\nu} = \frac{2\sqrt{K_m K_1}}{\sqrt{K_m K_1} + \sqrt{K_m K_2}} = \frac{2n_1}{n_1 + n_2} \quad (43)$$

where n_1 and n_2 are the optical indices of the spaces at $z < 0$ and $z > 0$, respectively [9]. In Fig. 1 we plot $\text{Re}(\bar{E}_x(z))$ in terms of z together with $K_e(z)$ for certain values of parameters. The smooth change in the dielectric constant and amplitude of the electric field are clear.

To complete our solution we rewrite the electric field together with the magnetic field of a plane wave solution which propagates in positive z direction:

$$\mathbf{E}(\mathbf{r}, t) = \hat{\mathbf{x}} E_{02} (-\xi)^{-\frac{i\nu}{2a}} (1-\xi)^\rho F(\alpha, \beta; \gamma; \xi) e^{i\omega t}, \quad (44)$$

$$\mathbf{B}(\mathbf{r}, t) = \hat{\mathbf{y}} \frac{iE_{02}}{2\omega a} (-\xi)^{-\frac{i\nu}{2a}} (1-\xi)^\rho \left[\frac{2a\alpha\beta}{\gamma} F(\beta+1, \alpha+1; \gamma+1; \xi) + \left(\frac{\nu i}{\xi} - \frac{2a\rho}{(1-\xi)} \right) F(\alpha, \beta; \gamma; \xi) \right] e^{i\omega t}. \quad (45)$$

IV. SMOOTH DOUBLE LAYERS

Another application of the general equation is to find the EMW passing through a double layer thick shell. For this let us consider $K_m = \text{const.}$ with

$$K_e(z) = K_1 + \frac{K_2 - K_1}{2} (\tanh az - \tanh a(z - L)). \quad (46)$$

Here a is a constant which in the limit $a \rightarrow \infty$, L becomes the thickness of a flat double layer dielectric of dielectric constant K_2 located inside another medium of dielectric constant K_1 . Next, we rewrite the wave equation (9) in this matter:

$$\left(\frac{d^2}{dz^2} + \left\{ \kappa^2 + \frac{\nu^2 - \kappa^2}{2} (\tanh az - \tanh a(z - L)) \right\} \right) \bar{E}_x(z) = 0, \quad (47)$$

in which κ and ν are defined in (15). We follow a similar change of variable given by (16) which modifies the latter equation as

$$\xi^2 E''(\xi) + \xi E'(\xi) + \frac{1}{4a^2} \left(\kappa^2 - \frac{(\nu^2 - \kappa^2)(\lambda - 1)}{\lambda(\xi - 1)(\xi - \frac{1}{\lambda})} \right) E(\xi) = 0 \quad (48)$$

in which $\lambda = \exp(2aL)$. Now, we replace $E(\xi) = \xi^\sigma H(\xi)$ with $\sigma = -i\kappa/2a$ to find

$$H''(\xi) + \frac{1 - i\kappa/a}{\xi} H'(\xi) + \frac{(\kappa^2 - \nu^2)(\lambda - 1)}{4a^2 \lambda \xi (\xi - 1)(\xi - \frac{1}{\lambda})} H(\xi) = 0 \quad (49)$$

which is a Heun differential equation [10] of the form

$$w''(z) + \left(\frac{\gamma}{z} + \frac{\delta}{z-1} + \frac{\epsilon}{z-p} \right) w'(z) + \frac{\alpha\beta z - q}{z(z-1)(z-p)} w(z) = 0 \quad (50)$$

in which $\epsilon = \alpha + \beta - \gamma - \delta + 1$, with a general solution

$$w(z) = C_1 \text{HeunG}(p, q, \alpha, \beta, \gamma, \delta, z) + C_2 z^{1-\gamma} \text{HeunG}(p, q - (p\delta + \epsilon)(\gamma - 1), \beta - \gamma + 1, \alpha - \gamma + 1, 2 - \gamma, \delta, z). \quad (51)$$

Herein C_1 and C_2 are two integration constants. Following the general solution we find the final solution of the wave equation as

$$\begin{aligned} E(\xi) = & C_1 (-\xi)^{\frac{-i\kappa}{2a}} \text{HeunG} \left(\frac{1}{\lambda e}, \frac{(\nu^2 - \kappa^2)(\lambda - 1)}{4a^2 \lambda}, 0, \frac{-i\kappa}{a}, \frac{a - i\kappa}{a}, 0, \xi \right) + \\ & C_2 (-\xi)^{\frac{i\kappa}{2a}} \text{HeunG} \left(\frac{1}{\lambda}, \frac{(\nu^2 - \kappa^2)(\lambda - 1)}{4a^2 \lambda}, 0, \frac{i\kappa}{a}, \frac{a + i\kappa}{a}, 0, \xi \right). \end{aligned} \quad (52)$$

Upon considering $\text{HeunG}(p, q, \alpha, \beta, \gamma, \delta, 0) = 1$, one easily finds that $\lim_{\xi \rightarrow 0} E(\xi) = C_1 \exp(i\kappa z) + C_2 \exp(-i\kappa z)$ which after assuming that the wave starts from $z = -\infty$ and propagates toward $z = +\infty$ one must set $C_2 = 0$ and $C_1 = E_{03}$. These, therefore, imply

$$E(\xi) = E_{03} (-\xi)^{\frac{-i\mu}{2a}} \text{HeunG} \left(\frac{1}{\lambda}, \frac{(\nu^2 - \kappa^2)(\lambda - 1)}{4a^2 \lambda}, 0, \frac{-i\mu}{a}, \frac{a - i\kappa}{a}, 0, \xi \right), \quad (53)$$

where E_{03} is the amplitude of the electric field at the limit $z \rightarrow +\infty$. Fig. 2 displays the continuous passage of EMW through the double layers located at $z = 0$ and $z = 4$.

V. CONCLUSION

In conclusion we have considered media with variable permeability and permittivity varying with position. The general form of the wave equation in such a non-uniform medium is presented. Some simplifications have been

considered such as, ϵ and μ are only functions of z which is also the direction of the propagation of the possible EMW. In such a framework we have used the sourceless Maxwell's equations to find a wave equation for the propagation of the EMW. We solved the wave equation for a smooth-step, variable dielectric given the constant permeability. In the limits we obtain the well known reflection and transmission coefficients of the plane waves in normal incidence on the interface plane between two dielectrics. Our results have been presented analytically and schematically. Furthermore one could in principle respect not only to the position dependent permeability and permittivity but also a time dependent form of them which may occur in chemical solutions while some reactions are presented therein. In anticipation of such a case the standard wave equation would modify in the form

$$\nabla^2 \mathbf{E} - \epsilon\mu \frac{\partial^2 \mathbf{E}}{\partial t^2} = (\mu\dot{\epsilon} + \dot{\mu}\epsilon) \frac{\partial \mathbf{E}}{\partial t} + (\dot{\mu}\dot{\epsilon} + \mu\ddot{\epsilon}) \mathbf{E} \quad (54)$$

and

$$\nabla^2 \mathbf{B} - \epsilon\mu \frac{\partial^2 \mathbf{B}}{\partial t^2} = \mu\dot{\epsilon} \frac{\partial \mathbf{B}}{\partial t} \quad (55)$$

in which an over dot means time derivative, $\epsilon = \epsilon(t)$ and $\mu = \mu(t)$. Studying this interesting problem is our future plan. The applications of such extremal theories may not be so clear yet but we believe that with the fast developments of the new detecting methods of cancerous cells or organs in biomedical optics, such detailed theories will contribute to have more accurate results. Finally, in the present work we have solved the smooth double layer problem in terms of the Heun's functions.

-
- [1] F. Fogolari, A. Brigo and H. Molinari, J. Mol. Recognit. **15**, 377 (2002).
 - [2] M. H. Andrews, A. H. Marshak and R. Shrivastava, J. Appl. Phys. **52**, 6783 (1981).
 - [3] M. V. Basilevsky, F. V. Grigoriev, E. A. Nikitina and J. Leszczynski, J. Phys. Chem. B, **114**, 2457 (2010).
 - [4] H. Nymeyer and H.-X. Zhou, Biophysical Journal, **94**, 1185 (2008).
 - [5] G. Le and J. Zhang, Langmuir, **27**, 5366 (2011).
 - [6] F. Fogolari, P. Zuccato, G. Esposito, P. Viglino, Biophysical Journal **76**, 1 (1999).
 - [7] I. Borukhov1, D. Andelmanb, H. Orland, Electrochimica Acta, **46**, 221 (2000).
 - [8] G. B. Arfken and H. J. Weber, Mathematical Methods for Physics, Harcourt Academic Press, Fifth Edition (2001).
 - [9] J. Jackson, Classical Electrodynamics, John Wiley & Sons, Third Edition, (1999).
 - [10] A. Ronveaux, Heun's Differential Equations, Oxford Science Publications, (1995).

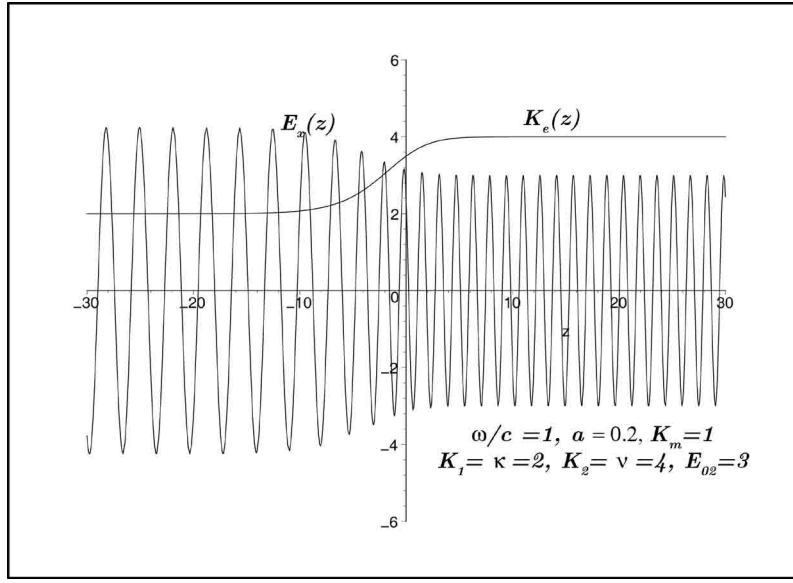


FIG. 1: A plot of $\text{Re}(\bar{E}_x(z))$ in terms of z together with $K_e(z)$ for certain values of the parameters. The smooth changes in the dielectric constant and amplitude of the electric field are clear.

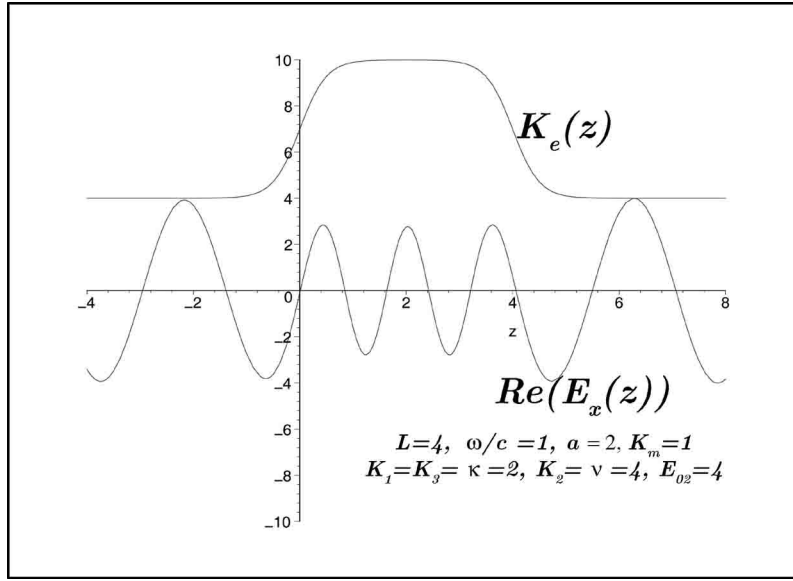


FIG. 2: The incoming EMW from $z < 0$ encounters with the first layer at $z = 0$. $\text{Re}(E_x(z))$ has similar structure after crossing the second layer at $z = 4$. In between, $0 < z < 4$, the oscillatory behaviour evidently changes.